Homework 4
MTH 829 Complex Analysis

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Proposition 0.1 (Exercise VI.8.1). *Let $z_1, z_2 \in \mathbb{C}$. Then for $n = 0, 1, 2, \ldots$ we have*

$$\int_{[z_1,z_2]} z^n dz = \frac{z_1^{n+1} - z_2^{n+1}}{n+1}$$
$$\int_{[z_1,z_2]} \overline{z}^n dz = \frac{(z_2 - z_1)(\overline{z}_2 - \overline{z}_1)^{n+1}}{n+1}$$

Proof. To compute the first integral, we note that $z \mapsto z^n$ has a primitive on all of $\mathbb{C}$, given by $F(z) = \frac{z^{n+1}}{n+1}$. Then

$$\int_{[z_1,z_2]} z^n dz = \int_{[z_1,z_2]} F' dz = F(z_1) - F(z_2) = \frac{z_1^{n+1} - z_2^{n+1}}{n+1}$$

Let $\gamma(t) = (1-t)z_1 + tz_2$. Then $\gamma'(t) = z_2 - z_1$, so applying the definition and using linearity, we get

$$\int_{[z_1,z_2]} \overline{z}^n dz = \int_0^1 \overline{\gamma(t)}^n (z_2 - z_1) dt = (z_2 - z_1) \int_0^1 \left((1-t)\overline{z}_1 + t\overline{z}_2\right)^n dt$$
$$= (z_2 - z_1) \int_0^1 (\overline{z}_1 + t(\overline{z}_2 - \overline{z}_1))^n dt$$

Now we perform a substitution. Let $u = t(\overline{z}_2 - \overline{z}_1)$. Then $du = (\overline{z}_2 - \overline{z}_1)dt$, so continuing to work with the same expression gives

$$(z_2 - z_1) \int_{t=0}^{t=1} \frac{u^n}{\overline{z}_2 - \overline{z}_1} du = \frac{z_2 - z_1}{\overline{z}_2 - \overline{z}_1} \int_{u=0}^{u=\overline{z}_2 - \overline{z}_1} u^n du = \frac{z_2 - z_1}{\overline{z}_2 - \overline{z}_1} \int_{[0,\overline{z}_2 - \overline{z}_1]} u^n du$$

Applying the computation above,

$$= \left(\frac{z_2 - z_1}{\overline{z}_2 - \overline{z}_1}\right) \frac{(\overline{z}_2 - \overline{z}_1)^{n+1}}{n+1} = \frac{(z_2 - z_1)(\overline{z}_2 - \overline{z}_1)^n}{n+1}$$

□

Proposition 0.2 (Exercise VI.8.3)**.** *Let f be a continuous complex valued function defined on $|z - z_0| < R$. For $0 < r < R$, let C_r denote the circle $|z - z_0| = r$. Then*

$$\lim_{r \to 0} \int_{C_r} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Proof. Parametrize C_r by $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + re^{it}$. Then $\gamma'(t) = ire^{it}$. Applying the definition of the integral,

$$\begin{aligned} \int_{C_r} \frac{f(z)}{z - z_0} dz &= \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z_0} ire^{it} dt = i \int_0^{2\pi} \frac{f(\gamma(t))}{re^{it}} re^{it} dt \\ &= i \int_0^{2\pi} f(\gamma(t)) dt = i \int_0^{2\pi} f(z_0 + re^{it}) dt \end{aligned}$$

Let $\epsilon > 0$. Since f is continuous, there exists $\delta > 0$ so that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

Choose $r < \delta$. Then

$$r = |re^{it}| = |(z_0 + re^{it}) - z_0| < \delta \implies |f(z_0 + re^{it}) - f(z_0)| < \epsilon$$

Then

$$\begin{aligned} \left| \int_{C_r} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| &= \left| i \int_0^{2\pi} f(z_0 + re^{it}) dt - 2\pi i f(z_0) \right| \\ &= \left| i \int_0^{2\pi} f(z_0 + re^{it}) dt - i \int_0^{2\pi} f(z_0) dt \right| = \left| i \int_0^{2\pi} \big(f(z_0 + re^{it}) - f(z_0) \big)\, dt \right| \\ &\le \int_0^{2\pi} \left| f(z_0 + re^{it}) - f(z_0) \right| dt \le \int_0^{2\pi} \epsilon \, dt = 2\pi i \epsilon \end{aligned}$$

Putting this all together, for $\epsilon > 0$, there exists $\delta > 0$ so that

$$r < \delta \implies \left| \int_{C_r} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| \le 2\pi i \epsilon$$

which is equivalent to

$$\lim_{r \to 0} \int_{C_r} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

□

Proposition 0.3 (Exercise VI.12.2)**.**

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \sqrt{\frac{\pi}{8}}$$

Proof. Let $R > 0$ and let $G_R = \left\{z : |z| \leq R, 0 \leq \operatorname{Arg} z \leq \frac{\pi}{4}\right\}$. Let γ_R be the boundary curve of G_R, oriented counterclockwise. Then γ_R is the "union" of three continuous curves $\gamma_R^1, \gamma_R^2, \gamma_R^3$ with $\gamma_R^1 = [0, R]$, $\gamma_R^3 = [Re^{i\pi/4}, 0]$, and $\gamma_R^2(t) = Re^{it\pi/4}$ for $t \in [0, 1]$. We parametrize γ_R^1, γ_R^3 in the standard way, $\gamma_R^1(t) = Rt$ and $\gamma_R^3(t) = (1-t)Re^{i\pi/4}$. The derivatives are

$$\gamma_R^1{}'(t) = R \qquad \gamma_R^2{}'(t) = \frac{Ri\pi}{4}e^{it\pi/4} \qquad \gamma_R^3{}'(t) = -Re^{i\pi/4}$$

Since the function $z \mapsto e^{-z^2}$ is holomorphic on $\mathbb{C}$,

$$0 = \int_{\gamma_R} e^{-z^2} dz = \sum_{j=1}^{3} \int_{\gamma_R^j} e^{-z^2} dz$$

We compute these three integrals individually.

$$\int_{\gamma_R^1} e^{-z^2} dz = \int_0^1 e^{-R^2t^2} R\, dt = \int_0^R e^{-x^2} dx$$

Sarason computes $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (page 72), and since the integrand is symmetric about the y-axis, taking the limit $R \to \infty$ we get

$$\lim_{R\to\infty} \int_{\gamma_R^1} e^{-z^2} dz = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Now for γ_R^3. After applying the definition, we make the substitutions $x = (1-t)R$ and $dx = -Rdt$ to get

$$\begin{aligned}
\int_{\gamma_R^3} e^{-z^2} dz &= \int_0^1 e^{-((1-t)Re^{i\pi/4})^2}(-Re^{i\pi/4})dt = \int_R^0 e^{i\pi/4} e^{-ix^2} dx = -e^{i\pi/4} \int_0^R e^{-ix^2} dx \\
&= -e^{i\pi/4} \int_0^R \cos(-x^2) + i \sin(-x^2) dx = -e^{i\pi/4} \int_0^R \cos(x^2) - i \sin(x^2) dx \\
&= -e^{i\pi/4} \int_0^R \cos(x^2) dx + ie^{i\pi/4} \int_0^R \sin(x^2) dx
\end{aligned}$$

Taking the limit as R goes to infinity,

$$\lim_{R\to\infty} \int_{\gamma_R^3} e^{-z^2} dz = -e^{i\pi/4} \int_0^{\infty} \cos(x^2) dx + ie^{i\pi/4} \int_0^{\infty} \sin(x^2) dx$$

So now if we can compute the integral over γ_R^2, we can write the integrals we set out to compute in terms of its real and imaginary parts.

$$\left| \int_{\gamma_R^2} e^{-z^2} dt \right| \leq \int_0^1 \left| e^{-(Re^{it\pi/4})^2} \right| \left| \frac{Ri\pi}{4} e^{it\pi/4} \right| dt = \int_0^1 \left| e^{-R^2 e^{it\pi/2}} \right| \left(\frac{R\pi}{4} \right) dt = \frac{R\pi}{4} \int_0^1 e^{-R^2 \cos\left(\frac{\pi}{2}t\right)} dt$$

Note that for $t \in [0,1]$, we have the inequality $1 - t \leq \cos\left(\frac{\pi}{2}t\right)$, so

$$e^{-R^2\cos\left(\frac{\pi}{2}t\right)} \leq e^{-R^2(1-t)} \implies \frac{R\pi}{4}\int_0^1 e^{-R^2\cos\left(\frac{\pi}{2}t\right)}dt \leq \frac{R\pi}{4}\int_0^1 e^{-R^2(1-t)}dt$$

Now in this integral we substitute $x = 1 - t$ and $dx = -dt$ to get

$$\frac{R\pi}{4}\int_0^1 e^{-R^2 x}dx = \frac{R\pi}{4}e^{-R^2}\int_0^1 e^x dx = \frac{R\pi}{4e^{R^2}}(e-1)$$

which goes to zero as $R \to \infty$. So combining our computations,

$$\begin{aligned}
&e^{i\pi/4}\int_0^\infty \cos(x^2)dx - ie^{i\pi/4}\int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2} \\
\implies &\int_0^\infty \cos(x^2)dx - i\int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2e^{i\pi/4}} = \sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}} \\
\implies &\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \sqrt{\frac{\pi}{8}}
\end{aligned}$$

□

Proposition 0.4 (Exercise VII.4.1)**.** *Let* $b \in (0,1)$*. Then*

$$\int_{-\infty}^\infty \frac{1-b+x^2}{(1-b+x^2)^2+4bx^2}dx = \pi$$

Proof. Let $a > 0$ and let R be the rectangle with vertices $\pm a, \pm a + i\sqrt{b}$. Let γ be the boundary of R oriented counterclockwise. Then $f(z) = \frac{1}{1+z^2}$ is holomorphic on R, since the denominator vanishes only at $\pm i$, and $b < 1 \implies \sqrt{b} < 1$ implies that $\pm i$ lie outside R. By Cauchy's Theorem for a convex region,

$$\int_\gamma \frac{1}{1+z^2}dz = 0$$

We can break this integral into four parts,

$$\int_\gamma \frac{1}{1+z^2}dz = \left(\int_{[a,a+i\sqrt{b}]} + \int_{[a+i\sqrt{b},-a+i\sqrt{b}]} + \int_{[-a+i\sqrt{b},-a]} + \int_{[-a,a]}\right) f dz$$

We evaluate each integrals individually. Let $\gamma_1 = [-a,a]$, parametrized as $\gamma_1(t) = a(2t-1)$ for $t \in [0,1]$. Then using the definition and the substitution $u = a(2t-1)$, we get

$$\int_{\gamma_1} f dz = \int_0^1 \frac{1}{1+(a(2t-1))^2}2a\,dt = \int_{-a}^a \frac{1}{1+u^2}du = \arctan\Big|_{-a}^a = 2\arctan a$$

Note that

$$\lim_{a\to\infty}\int_{\gamma_1} f dz = \lim_{a\to\infty} 2\arctan a = \pi$$

Let $\gamma_2 = [a, a + i\sqrt{b}]$, parametrized as $\gamma_2(t) = ti\sqrt{b} + a$. Then applying the definition and making the subtitution $u = t\sqrt{b}$,

$$\int_{\gamma_2} f dz = \int_0^1 \frac{1}{1 + (ti\sqrt{b} + a)^2} i\sqrt{b}\, dt = i \int_0^{\sqrt{b}} \frac{1}{1 + (iu + a)^2} du$$

Multiplying out the denominator and multiplying by the conjugate, we get

$$\begin{aligned}\frac{1}{1 + (iu + a)^2} &= \frac{1}{1 + a^2 - u^2 + 2aui} = \frac{1}{1 + a^2 - u^2 + 2aui} \left(\frac{1 + a^2 - u^2 - 2aui}{1 + a^2 - u^2 - 2aui} \right) \\ &= \frac{1 + a^2 - u^2 - 2aui}{(1 + a^2 - u^2)^2 + 4a^2u^2}\end{aligned}$$

Let $\gamma_4 = [-a + i\sqrt{b}, -a]$, parametrized as $\gamma_4(t) = i\sqrt{b}(1-t) - a$. Then applying the definition and making the substitution $u = \sqrt{b}(1 - t)$, we get

$$\int_{\gamma_4} f dz = \int_0^1 \frac{1}{1 + (i\sqrt{b}(1 - t) - a)^2} (-i\sqrt{b}) dt = i \int_{\sqrt{b}}^0 \frac{1}{1 + (iu - a)^2} du$$

Multiplying out and multiplying by the conjugate as above,

$$\begin{aligned}\frac{1}{1 + (iu - a)^2} &= \frac{1}{1 + a^2 - u^2 - 2aui} = \frac{1}{1 + a^2 - u^2 - 2aui} \left(\frac{1 + a^2 - u^2 + 2aui}{1 + a^2 - u^2 + 2aui} \right) \\ &= \frac{1 + a^2 - u^2 + 2aui}{(1 + a^2 - u^2)^2 + 4a^2u^2}\end{aligned}$$

So we get

$$\begin{aligned}\int_{\gamma_2} f dz &= i \int_0^{\sqrt{b}} \frac{1 + a^2 - u^2 - 2aui}{(1 + a^2 - u^2)^2 + 4a^2u^2} du \\ \int_{\gamma_4} f dz &= -i \int_0^{\sqrt{b}} \frac{1 + a^2 - u^2 + 2aui}{(1 + a^2 - u^2)^2 + 4a^2u^2} du\end{aligned}$$

Adding these together, we see that the real parts inside the integral cancel,

$$\begin{aligned}\int_{\gamma_2} f dz + \int_{\gamma_4} f dz &= i \int_0^{\sqrt{b}} \frac{1 + a^2 - u^2 - 2aui}{(1 + a^2 - u^2)^2 + 4a^2u^2} du - i \int_0^{\sqrt{b}} \frac{1 + a^2 - u^2 + 2aui}{(1 + a^2 - u^2)^2 + 4a^2u^2} du \\ &= i \int_0^{\sqrt{b}} \frac{4au}{(1 + a^2 - u^2)^2 + 4a^2u^2} du\end{aligned}$$

Using mathematica, this integral is

$$i \arctan \left(\frac{a^2 + u^2 - 1}{2a} \right) \Bigg|_{u=0}^{u=\sqrt{b}} = i \arctan \left(\frac{a^2 + b - 1}{2a} \right) - i \arctan \left(\frac{a^2 - 1}{2a} \right)$$

Note that

$$\lim_{a\to\infty} \arctan\left(\frac{a^2+b-1}{2a}\right) = \lim_{a\to\infty} \arctan\left(\frac{a^2-1}{2a}\right) = \frac{\pi}{2}$$

so

$$\lim_{a\to\infty}\left(\int_{\gamma_2} f dz + \int_{\gamma_4} f dz\right) = 0$$

Now let $\gamma_3 = [a+i\sqrt{b}, -a+i\sqrt{b}]$ and parametrize it by $\gamma_3(t) = a(1-2t)+i\sqrt{b}$. Then applying the definition and making the substitution $u = a(1-2t$,

$$\int_{\gamma_3} f dz = \int_0^1 \frac{1}{1+(a(1-2t)+i\sqrt{b})^2}(-2a)dt = \int_a^{-a} \frac{1}{1+(u+i\sqrt{b})^2}du$$

Working with the integrand, we simplify it to

$$\begin{aligned}\frac{1}{1+(u+i\sqrt{b})^2} &= \frac{1}{1+u^2-b+2ui\sqrt{b}} = \frac{1}{1+u^2-b+2ui\sqrt{b}}\left(\frac{1+u^2-b-2ui\sqrt{b}}{1+u^2-b-2ui\sqrt{b}}\right)\\ &= \frac{1+u^2-b-2ui\sqrt{b}}{(1+u^2-b)^2+4u^2b}\end{aligned}$$

So we can rewrite our integral as

$$\int_{\gamma_3} f dz = \int_a^{-a} \frac{1+u^2-b}{(1+u^2-b)^2+4u^2b}du + i\int_a^{-a} \frac{-2u\sqrt{b}}{(1+u^2-b)^2+4u^2b}du$$

Using mathematica, we evaluate the imaginary part as

$$\begin{aligned}\int_a^{-a} \frac{-2u\sqrt{b}}{(1+u^2-b)^2+4u^2b}du &= \frac{1}{2}\tanh^{-1}\left(\frac{u^2+b+1}{2\sqrt{b}}\right)\Bigg|_{-a}^{a}\\ &= \frac{1}{2}\tanh^{-1}\left(\frac{a^2+b+1}{2\sqrt{b}}\right) - \frac{1}{2}\tanh^{-1}\left(\frac{a^2+b+1}{2\sqrt{b}}\right)\\ &= 0\end{aligned}$$

So we get

$$\int_{\gamma_3} f dz = -\int_{-a}^{a} \frac{1+u^2-b}{(1+u^2-b)^2+4u^2b}du$$

Combining all of our results,

$$\begin{aligned}\left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3}+\int_{\gamma_4}\right) f dz = 0 &\implies \lim_{a\to\infty}\left(\int_{\gamma_1}+\int_{\gamma_3}\right) f dz = 0\\ &\implies \lim_{a\to\infty} -\int_{\gamma_3} f dz = \lim_{a\to\infty}\int_{\gamma_1} f dz\\ &\implies \lim_{a\to\infty}\int_{-a}^{a} \frac{1-b+x^2}{(1-b+x^2)^2+4bx^2}dx = \lim_{a\to\infty} 2\arctan a\\ &\implies \int_{-\infty}^{\infty} \frac{1-b+x^2}{(1-b+x^2)^2+4bx^2}dx = \pi\end{aligned}$$

$\square$

Proposition 0.5 (Exercise VII.5.1). *Let $G \subset \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be holomorphic. Let C be a counterclockwise oriented circle with $C \subset G$ and the interior of C contained in G. Then for z_0 in G in the exterior of C,*

$$\int_C \frac{f(z)}{z - z_0} dz = 0$$

Proof. Define $g : G \setminus \{z_0\} \to \mathbb{C}$ by $g(z) = \frac{f(z)}{z-z_0}$. Then g is holomorphic on $G \setminus \{z_0\}$ in a convex neighborhood of C, so by Cauchy's Theorem for a convex region (VII.2 of Sarason),

$$\int_C g(z) dz = \int_C \frac{f(z)}{z - z_0} dz = 0$$

□

Proposition 0.6 (Exercise VII.6.1). *Let $z_0 \in \mathbb{C}$ and $R > 0$. Let f be holomorphic in the disk $D = \{z : |z - z_0| < R\}$. Then for $0 < r < R$, the value of f at z_0 is the average of f over the disk $|z - z_0| < r$. That is,*

$$f(z_0) = \frac{1}{\pi r^2} \iint_D f(z) dA$$

Proof. This pretty much just a computation, involving an application of change of variables and Fubini's theorem.

$$\frac{1}{\pi r^2} \iint_D f(z) dA = \frac{1}{\pi r^2} \iint_D f(x + iy) dx\, dy = \frac{1}{\pi r^2} \iint_D f(z_0 + se^{i\theta}) s\, ds\, d\theta$$

where the extra factor of s comes from the Jacobian determinant in the change to polar coordinates. Now we apply Fubini's Theorem to change the order of integration.

$$= \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r f(z_0 + se^{i\theta}) s\, ds\, d\theta = \frac{1}{\pi r^2} \int_0^r s \left(\int_0^{2\pi} f(z_0 + se^{i\theta}) d\theta \right) ds$$

Applying the result for averages over circles,

$$= \frac{1}{\pi r^2} \int_0^r s(2\pi f(z_0)) ds = \frac{2\pi}{\pi r^2} f(z_0) \int_0^r s\, ds = \frac{2}{r^2} f(z_0) \frac{r^2}{2} = f(z_0)$$

□

Proposition 0.7 (Exercise VII.7.1). *The Cauchy integral of the constant function 1 over $[0, 1]$ is*

$$f(x + iy) = \frac{1}{2} \ln \left(\frac{(1 - x)^2 + y^2}{x^2 + y^2} \right) + i \left(\arctan \left(\frac{1 - x}{y} \right) + \arctan \left(\frac{x}{y} \right) \right)$$

Proof. Let $\gamma = [0,1]$, parametrized by $\gamma(t) = t$. Let $z = x + iy$ with $x, y \in \mathbb{R}$. Then

$$\begin{aligned}\int_\gamma \frac{1}{w-z}dw &= \int_0^1 \frac{1}{t-x-iy}dt = \int_0^1 \frac{1}{t-x-iy}\left(\frac{t-x+iy}{t-x+iy}\right)dt = \int_0^1 \frac{t-x+iy}{(t-x)^2+y^2}dt \\ &= \int_0^1 \frac{t-x}{(t-x)^2+y^2}dt + i\int_0^1 \frac{y}{(t-x)^2+y^2}dt\end{aligned}$$

First we compute the real part by making the substitution $u = (t-x)^2 + y^2$ and $du = 2(t-x)dt$.

$$\begin{aligned}\int_0^1 \frac{t-x}{(t-x)^2+y^2}dt &= \int_{x^2+y^2}^{(1-x)^2+y^2} \frac{1}{2u}du = \frac{1}{2}\ln|u|\Big|_{x^2+y^2}^{(1-x)^2+y^2} \\ &= \frac{1}{2}\ln((1-x)^2+y^2) - \frac{1}{2}\ln(x^2+y^2) = \frac{1}{2}\ln\left(\frac{(1-x)^2+y^2}{x^2+y^2}\right)\end{aligned}$$

Note that $x + iy = 0$ and $x + iy = 1$ are not in our domain, and the square of a real number is nonnegative, so this always makes sense. Now we compute the imaginary part, making the substitution $u = t - x$ and $du = dt$.

$$\begin{aligned}\int_0^1 \frac{y}{(t-x)^2+y^2}dt &= \arctan\left(\frac{t-x}{y}\right)\Big|_{t=0}^{t=1} = \arctan\left(\frac{1-x}{y}\right) - \arctan\left(\frac{-x}{y}\right) \\ &= \arctan\left(\frac{1-x}{y}\right) + \arctan\left(\frac{x}{y}\right)\end{aligned}$$

Thus

$$\int_{[0,1]} \frac{1}{w-z}dw = \frac{1}{2}\ln\left(\frac{(1-x)^2+y^2}{x^2+y^2}\right) + i\left(\arctan\left(\frac{1-x}{y}\right) + \arctan\left(\frac{x}{y}\right)\right)$$

□

Proposition 0.8 (Exercise VII.8.1a)**.** *Let C be the unit circle. Then*

$$\int_C \frac{\sin z}{z^{38}}dz = \frac{2\pi i}{37!}$$

Proof. By the Cauchy integral formula,

$$\frac{d^{37}}{dw^{37}}\sin w = \frac{37!}{2\pi i}\int_C \frac{\sin z}{(z-w)^{38}}dz$$

The 37th derivative of $\sin w$ is $\cos w$. Setting $w = 0$,

$$\cos 0 = \frac{37!}{2\pi i}\int_C \frac{\sin z}{z^{38}}dz \implies \int_C \frac{\sin z}{z^{38}}dz = \frac{2\pi i}{37!}$$

□

Proposition 0.9 (Exercise VII.8.2)**.**

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-2r\cos\theta+r^2}d\theta=\frac{1}{1-r^2}$$

Proof. Fix $r\in(0,1)$. First we rewrite the integral to look nicer.

$$\begin{aligned}\frac{1}{1-2r\cos\theta+r^2}&=\frac{1}{1-2r\left(\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right)+r^2}=\frac{1}{1-re^{i\theta}-re^{-i\theta}+r^2}\\&=\frac{e^{i\theta}}{e^{i\theta}-re^{i2\theta}-r+r^2e^{i\theta}}=\frac{e^{i\theta}}{(e^{i\theta}-r)(1-re^{i\theta})}\end{aligned}$$

Now define

$$f(z)=\frac{1}{i(1-rz)}$$

Note that f is holomorphic except at $z=\frac{1}{r}$. Parametrize the unit circle C by $\gamma(\theta)=e^{i\theta}$ for $\theta\in[0,2\pi]$. Then $\gamma'(\theta)=ie^{i\theta}$, so

$$\int_C\frac{f(z)}{z-r}dz=\int_0^{2\pi}\frac{f(\gamma(\theta))\gamma'(\theta)}{e^{i\theta}-r}d\theta=\int_0^{2\pi}\frac{f(e^{i\theta})ie^{i\theta}}{e^{i\theta}-r}d\theta=\int_0^{2\pi}\frac{e^{i\theta}}{(e^{i\theta}-r)(1-re^{i\theta})}d\theta$$

By the Cauchy integral formula,

$$\int_C\frac{f(z)}{z-r}dz=2\pi if(r)=2\pi i\left(\frac{1}{i(1-r^2)}\right)=\frac{2\pi}{1-r^2}$$

Thus

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-2r\cos\theta+r^2}d\theta&=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}}{(e^{i\theta}-r)(1-re^{i\theta})}d\theta\\&=\frac{1}{2\pi}\int_C\frac{f(z)}{z-r}dz\\&=\frac{1}{2\pi}\left(\frac{2\pi}{1-r^2}\right)\\&=\frac{1}{1-r^2}\end{aligned}$$

□

Proposition 0.10 (Exercise VII.8.3)**.** *Let $a,b\in\mathbb{C}$with $|a|<1<|b|$. Let C be the unit circle with counterclockwise orientation. Let $m,n\in\mathbb{Z}$ with $n\geq1$. Then*

$$\frac{1}{2\pi i}\int_C\frac{(z-b)^m}{(z-a)^n}dz=\begin{cases}\frac{m!}{(m-n+1)!(n-1)!}(a-b)^{m-n+1} & n-1<m\\0 & n-1\geq m\end{cases}$$

Proof. Let $f(z)=(z-b)^m$. Then by the Cauchy integral formula,

$$f^{(n-1)}(a)=\frac{(n-1)!}{2\pi i}\int_C\frac{(z-b)^m}{(z-a)^n}dz\implies\frac{1}{2\pi i}\int_C\frac{(z-b)^m}{(z-a)^n}dz=\frac{f^{(n-1)}(a)}{(n-1)!}$$

(Note that here we use the fact that $n \geq 1$). So we just need to compute $f^{(n-1)}(a)$. If $n-1 \geq m$, then $f^{(n-1)}(a) = 0$. If $n-1 < m$, then

$$\begin{aligned}
f^{(1)}(z) &= m(z-b)^{m-1} \\
f^{(2)}(z) &= m(m-1)(z-b)^{m-2} \\
f^{(3)}(z) &= m(m-1)(m-2)(z-b)^{m-3} \\
&\vdots \\
f^{(n-1)}(z) &= m(m-1)(m-2)\dots(m-2+2)(z-b)^{m-n+1} \\
&= \frac{m!}{(m-n+1)!}(z-b)^{m-n+1}
\end{aligned}$$

So

$$f^{(n-1)}(a) = \frac{m!}{(m-n+1)!}(a-b)^{m-n+1}$$

Thus

$$\frac{1}{2\pi i}\int_C \frac{(z-b)^m}{(z-a)^n}dz = \begin{cases} \frac{m!}{(m-n+1)!(n-1)!}(a-b)^{m-n+1} & n-1 < m \\ 0 & n-1 \geq m \end{cases}$$

□